Dynamics of a Multibody System with Relative Translation on Curved, Flexible Tracks

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Previous generic formulations of equations of motion for multibody systems treat explicitly only special cases of interbody translation, such as unconstrained translation or translation constrained to a straight, rigid track or a rigid plane. But real, physical tracks are not always even nominally straight, and they are always somewhat flexible. In this paper, the previous formulations are extended to accommodate interbody translations that can be characterized nominally by a single scalar variable (such as distance traveled on a curved track or screw path) plus motions induced by small deformations of the track or guideway.

Introduction

THE multibody dynamics literature stimulated by spacecraft control problems has focused on relative rotations and deformations of bodies, because these have been the motions of greatest practical concern. As spacecraft continue to evolve, problems of interbody translation have come to the fore.

The manipulator arm on the Space Shuttle is a familiar example of a flexible mechanism with articulated joints rotated through large angles by means of active control devices. When a similar manipulator arm is considered for use on the much larger space station, however, it is often conceived as mounted on a small cart that moves along tracks or guideways attached to the space station structure. A track or guideway need not be straight, and its flexibility may be functionally significant.

Many different formulations of equations of motion for multiple, flexible body systems have been published (for a few of many possible examples see Refs. 1-3) and corresponding computer programs for simulation of generic cases or classes of spacecraft have been made available (such as DISCOS[2] and TREETOPS[1] in the public domain). The most general of these generic formulations permits interbody translation, but the published work is sufficient only for application to relatively simple illustrations.‡ In at least one paper,[1] the authors underestimated the difficulties of the general case and overstated the range of applicability of their work. That deficiency can now be rectified.

Context

The equations of motion of a system of interconnected flexible bodies are extremely complex, even when the details of interbody translation are omitted. Those equations are, however, in the public literature, and even computer codes required for simulations are in the public domain. In what follows, therefore, the contextual setting of Ref. 1 will be assumed, with special reference to published material taking the place of lengthy and complicated equations that would otherwise be required to make the results of this paper useful.

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‡Work in progress by J. Wittenburg as described at the IUTAM/IFTOMM Symposium on Multibody Dynamics in Udine, Italy, in Sept. 1985, will extend the field significantly.

A brief description of this context may be helpful. A succession of derivations using Kane's variant of Lagrange's form of D'Alembert's Principle by Singh and Likins[1,4,5] has produced a series of simulation programs by VanderVoort and Singh.[6] The equations of motion in Ref. 1 are restricted to a system of elastic bodies interconnected in a topological tree, and in Ref. 5 a method is advanced for eliminating constraints introduced by violating the tree topology restriction, closing loops of bodies. In both Refs. 1 and 5, relative translations are permitted between pairs of interconnected bodies, with arbitrary vector y^j locating point h_j of body j relative to point p_j fixed in the inboard adjacent body $c(j)$. [Most of the symbols are quoted from Ref. 1, but, for abbreviation, the superscripts (e.g., j) will generally be omitted in this paper.] The vector y in Ref. 1 is written in terms of a vector base (g_1, g_2, g_3) fixed in a reference frame b_{pj} embedded in body $c(j)$ at p_j. The velocity of h_j relative to p_j in frame b_{pj} is given by $(g_n \equiv 0)$

$$\overset{\circ}{y} = \sum_{n=1}^{NT_j} \dot{y}_n g_n \tag{1}$$

where NT_j is the number of translational degrees of freedom of the jth hinge. The text of Ref. 1 suggests that $\dot{y}_n$ can be chosen as a generalized speed in formulating the equations of motion, without acknowledging the restrictions implied by this selection. In fact, this choice is possible only if one of the following conditions applies: 1) h_j is constrained to move on a straight, rigid track fixed in b_{pj} at p_j; 2) h_j is constrained to move on a rigid plane similarly fixed; or 3) h_j is free to move without kinematic constraint. These restrictions preclude for example applications in which h_j is constrained to move on a curved rigid track fixed in b_{pj} or on a track that deforms as a part of flexible body $c(j)$. It is to these extensions that the present paper is devoted.

Problem Description

Equations of motion are derived in Ref. 1 for holonomic systems with NS degrees of freedom in terms of the equation

$$f_p + f_p^* = 0, \quad p = 1, \ldots, NS \tag{2}$$

where the generalized active forces f_p and the generalized inertial forces f_p^* are expressed, respectively, in Eqs. (14) and (15) of Ref. 1 [or Eqs. (38) and (36) in this paper] in terms of the coefficients of generalized speeds that Kane refers to as "par-

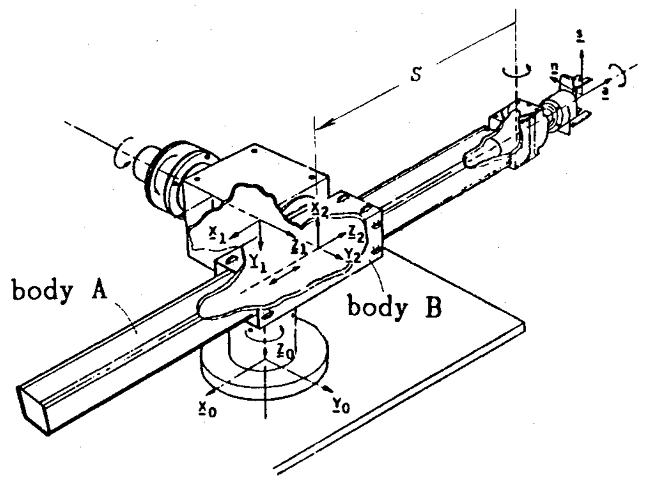

Fig. 1 Extendable arm, straight track.

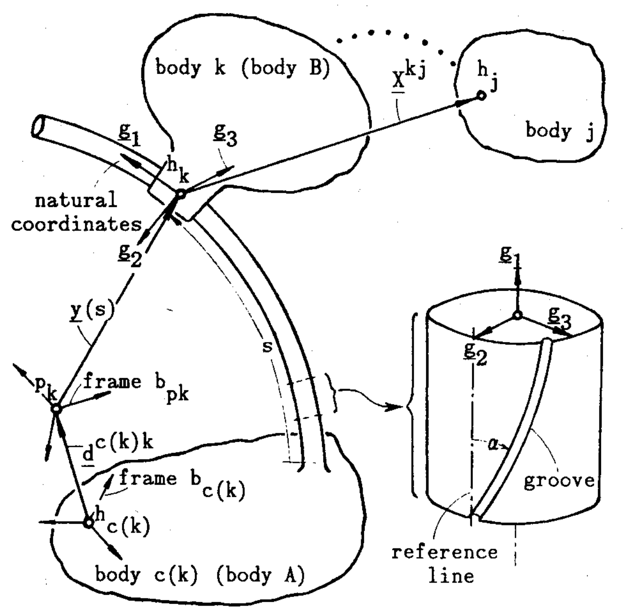

Fig. 2 Curved track on inboard body and an enlarged section of grooved track.

tial velocities."[7] The objective of the present paper is to obtain for substitution into these equations a set of "partial velocities" appropriate for a class of problems in which two bodies experience relative translation (and possible rotation) by the progression of one body along a curved, flexible track that constitutes a part of the other. In order to do so, a vector function $y(s)$ to describe the configuration of the undeformed track and a set of deformation modal functions are recommended. Those "partial velocities" and the contributions of the track to f_p and f_p^* are given in terms involving these functions.

Figure 1 portrays an extendable arm that provides a special (rectilinear) case of such a system, and Fig. 2 illustrates a more general case. In both figures, the motion of body B along the track of body A is defined by a single scalar s, which measures the distance traversed along the track. Additional motion of the system may result from the flexibility of both B and A, or by the overall translation or rotation of the total system.

Since the hinges shown in Figs. 1 and 2 are not topologically symmetric, which body, A or B, is on the inboard side (near the reference body 0, see Ref. 1) matters. First, we will treat the case in which the track is on the inboard body, then a method to deal with the opposite situation will be given. In the former case, we will first do the rigid simulation, then the influence of the flexibility will be considered.

Curved, Rigid Track

In this section, we will treat a hinge that connects body $c(k)$ and body k, and only has one degree of translational freedom. The track (labeled A) is a part of inboard body $c(k)$, and both are supposed to be rigid. Body B is connected with the track by a muff fitted over it. It should be assumed that the radius of the cross section and the length of the muff are very small as compared with the radius of curvature of the track centerline, and the track is in contact only with the two ends of the muff. If the cross section is a square, as in Fig. 1, the hinge has only one degree of freedom. If the cross section is a circle, as in Fig. 2, then there exists an additional rotational freedom around the track. To remove this freedom, a screw groove should be made on the surface of the bar, and a key should be fitted in the groove to guide the motion of the muff. Therefore, in this degree of freedom, body B slides along the track and, in the meantime, rotates around the track, but its position is solely determined by one variable s, the arc length, measured from a reference point on the track.

For convenience, the cross section will be assumed as a circle with radius R. In order to define the screw angle of the groove $\alpha(s)$, a reference line must be set up on the surface of the track as an initial line with respect to which to measure angle α. When the track is straight, any straight line on the surface parallel to the centerline can be referred to as a reference line. The screw angle α is measured from this line to the groove line, and α is positive when the groove line turns with the fingers of the right hand while the thumb points to the advanced direction of the muff. It is not difficult to find the relation between the angular and linear velocities of the muff

$$\dot{\theta} = [(\tan\alpha)/R]\dot{s} \triangleq \nu\dot{s} \tag{3}$$

where $\nu(s) \triangleq (\tan\alpha)/R$.

When the track is curved arbitrarily in space, there is no reference line that can be imagined easily. Let us put a set of unit base vectors (g_1, g_2, g_3) of natural coordinates on every point of the centerline of the track. The principal-normal g_2 penetrates the surface at a point. Connecting such points on the surface of the track will form a line. We define it as the reference line.

A typical sliding hinge (track on the inboard body) and its topological relation with related bodies are shown on Fig. 2. Point $h_{c(k)}$ is the link point connecting to another inboard body; it is also the origin of frame $b_{c(k)}$. We may describe the configuration of the track by a vector starting from $h_{c(k)}$, but usually to select another point p_k as the origin is more convenient for this purpose. Point p_k is fixed on frame $b_{c(k)}$, but not necessarily within the range of body $c(k)$. The configuration of the centerline of the track is determined by a vector function $y(s)$ starting from p_k. For example, when the track is a straight one or a part of a circle, we may select p_k right on the track or on the center of the circle, which enables the function $y(s)$ to be expressed in the most simple form.

Once the vector function $y(s)$ has been given, the unit base vectors of natural coordinates of the track centerline can easily be derived:

$$\begin{aligned} g_1 &= \mathrm{d}y/\mathrm{d}s \triangleq y'(s) && \text{tangent} \\ g_2 &= y''/\kappa && \text{principal-normal} \\ g_3 &= g_1 \times g_2 && \text{sub-normal} \end{aligned} \tag{4}$$

where $\kappa \triangleq |y''|$ is the curvature of the centerline of the track, and primes mean differentiation with respect to s. When the track is straight, κ will be zero, and the above definition of g_2 fails. In this case, we define a direction for g_2 arbitrarily.

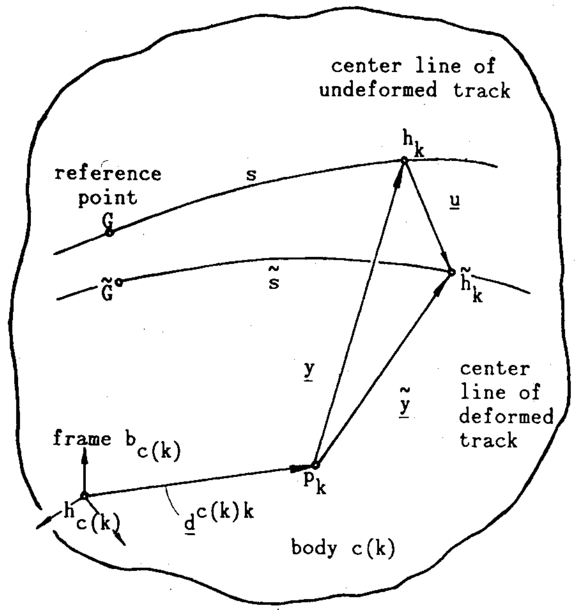

Fig. 3 Geometric relations between the deformed and undeformed track.

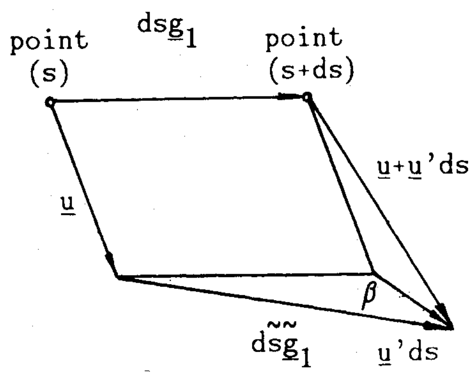

Fig. 4 Differential deformational displacements of two adjacent points on the track centerline.

Now we derive the important kinematic quantities of the hinge. First, the velocity of h_k in frame $b_{c(k)}$ is given by

$$V^{h_k} \triangleq \mathring{y} = y'\dot{s} = g_1\dot{s} \tag{5}$$

A dot means differentiation with respect to time; a circle on a vector means the differentiation is done in frame $b_{c(k)}$.

Second, the angular velocity ω of the vector base g_i $(i=1,2,3)$ in frame $b_{c(k)}$, as it slides along the track with its origin h_k, follows from the differentiation of g_i with respect to time in frame $b_{c(k)}$

$$\mathring{g}_1 = (y')^\circ = y''\dot{s} = \kappa\dot{s}g_2$$

in combination with

$$\mathring{g}_2 = \omega_1 g_3 - \omega_3 g_1 = (y''\cdot y''y''' - y''\cdot y'''y'')\dot{s}/\kappa^3$$

From the preceding two equations, we obtain

$$\omega_2 = 0 \tag{6}$$

$$\omega_3 = \kappa\dot{s} \tag{7}$$

Another differentiation provides

$$\mathring{g}_2 = (y''/\kappa)^\circ = (\kappa y'''\dot{s} - \dot{\kappa}y'')/\kappa^2$$

From $\kappa^2 = y''\cdot y''$, we have $\dot{\kappa} = y''\cdot y'''\dot{s}/\kappa$, so that

$$\mathring{g}_2 = \omega_1 g_3 - \omega_3 g_1 = (y''\cdot y''y''' - y''\cdot y'''y'')\dot{s}/\kappa$$

$$= (g_2\cdot g_2 y''' - g_2\cdot y''' g_2)\dot{s}/\kappa = (y'''\cdot g_1 g_1 + y'''\cdot g_3 g_3)\dot{s}/\kappa$$

and we obtain

$$\omega_1 = y'''\cdot g_3\dot{s}/\kappa = y'''\cdot y'\times y''\dot{s}/\kappa^2 \triangleq \tau\dot{s} \tag{8}$$

here $\tau \triangleq y'''\cdot y''/\kappa^2$ is the torsion of the curve; for a plane curve; $\tau = 0$. We also have another expression for ω_3: $\omega_3 = -y'''\cdot y'\dot{s}/\kappa$, which can easily be verified as the same as Eq. (7), if we note the identity $y'\cdot y'' \equiv 0$, and $(y'\cdot y'')'$ $y'\cdot y''' + y''\cdot y'' = 0$.

The third requirement is the angular velocity of frame b_k of the muff $\omega^{kc(k)}$ in frame $b_{c(k)}$. If the screw angle of the groove $\alpha = 0$, as in Fig. 1, g_i will be fixed on the muff, so $\omega^{kc(k)} = \omega$. If not so, as in Fig. 2, a term $\dot{\theta} = \nu\dot{s}$ should be added to the first component ω_1, then

$$\omega^{kc(k)} = [(\tau+\nu)g_1 + \kappa g_3]\dot{s} \tag{9}$$

Although the hinge discussed previously has one degree of freedom only, additional rotational freedom can easily be added by installing a gimbal joint between the small muff (now considered to be massless) and the body k. Then

$$\omega^{kc(k)} = [(\tau+\nu)g_1 + \kappa g_3]\dot{s} + \sum_{m=1}^{NR_k} \dot{\theta}_m l_m$$

where l_m represents unit vectors along gimbal axes, and NR_k is the number of rotational degrees of freedom of the kth hinge. Note $\omega^{kc(k)}$ is now divided into two terms; the first term is induced by translational freedom and the second term is due to the rotating movement of the joint between body k and its massless muff. This division is necessary because the torques in the joint are related to the above $\dot{\theta}_m$ and the corresponding Euler angles of the joint.

Finally, if $\dot{s}$ is selected as the pth generalized speed of the system, the corresponding partial angular velocity of the jth body and the partial velocity of point h_j will be [see Eqs. (5) and (9)]

$$\omega_p^j = \begin{cases} (\tau+\nu)y' + y''\times y''' & \text{for } j\epsilon P(k) \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

$$V_p^{h_j} = \begin{cases} y' + \omega_p^j \times X^{kj} & \text{for } j\epsilon P(k) \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

where $P(k)$ is a set of all bodies outboard of k including k, and X^{kj} is the vector leading from h_k to h_j at the present moment.

Track Flexibility

A sliding hinge is a fitting mechanism. The deformation models selected should not destroy the kinematic characteristics of the hinge, so that after deformation the muff should still be allowed to slide along the track freely. A reasonable modeling assumption is:

1) Every cross section can be treated as a rigid thin slice.

2) When the track deforms, every cross section is always perpendicular to the tangent of the centerline of the track.

The second assumption implies that we neglect the influence of shear of the track, and indeed shear deformations are generally negligible for such structural members. However, the influence of torsional deformations of the track may be important in the simulation of the motion of the system, particularly if the track is grooved.

According to the preceding assumptions, the modeling of the track can fully be determined by the following modal functions:

$$\psi_n(s) \text{ and } \Theta_n(s); \quad n=1,2,\ldots,NM_k$$

where vector $\psi_n(s)$ represents that part of the body modal deformation vector $\phi_n(r)$ (see Ref. 1) descriptive of the track centerline in mode n, and scalar $\Theta_n(s)$ represents the torsional model of the track in mode n; and NM_k is the number of the deformational degrees of freedom of the kth body. The deformational displacement of every point on the track centerline can be expressed as

$$u(s,t)=\sum_{n=1}^{NM_k}\psi_n(s)\eta_n(t) \tag{12}$$

The torsional angle of every cross section can be expressed as

$$\theta(s,t)=\sum_{n=1}^{NM_k}\Theta_n(s)\eta_n(t) \tag{13}$$

For convenience, the lower and upper limit of the summation will be omitted whenever it is self-evident. For example, the following symbols will be used often:

$$u'=\Sigma\psi'\eta, \quad u''=\Sigma\psi''\eta, \quad \theta'=\Sigma\Theta'\eta$$

In preparation for deriving partial (angular) velocities, we must do two things: 1) find the deformed positions of g_i that are embedded in the track, and 2) find the deformation displacement of any point on the track. Since our approach to deal with the problem of flexibility is a method of linearization based on the small deformation assumption that the generalized coordinates η_n and their derivatives are small quantities, so are u, θ and their derivatives; all the second-order terms in these variables can be neglected.

We still use the symbol $y(s)$ to describe the configuration of the centerline of undeformed track. We will add a tilde on a letter to indicate that it is a quantity after deformation. So $\tilde{y}$ is used to describe the deformed centerline (see Fig. 3):

$$\tilde{y}(s,t)=y(s)+u(s,t)=y(s)+\Sigma\psi(s)\eta(t) \tag{14}$$

After deformation, the arc length of the centerline is changed, s changes to $\tilde{s}$ (all measured in the same unit, such as cm). Now we study the elongation rate of the centerline (see Fig. 4)

$$e(s,t)=\frac{d\tilde{s}}{ds}-1$$

From the theorem of cosines, after some derivation,

$$e=\sqrt{1+2g_1\cdot u'+u'\cdot u'}-1 \tag{15}$$

Omitting the second-order terms of u', we have the linearized e:

$$e \simeq g_1\cdot u' \triangleq u_1' \tag{16}$$

Note that e is also a small quantity and $u_1'=(u')_{;1}\neq(u_1)'$.

The deformed positions of g_i will be written as $\tilde{g}_i$. First, let us find $\tilde{g}_1$. According to our assumption of perpendicularity, $\tilde{g}_1$ will still be along the tangent of the deformed centerline, so

$$\tilde{g}_1=\frac{\partial\tilde{y}}{\partial\tilde{s}}=\tilde{y}'\frac{ds}{d\tilde{s}}=(y'+u')/(1+e)\simeq(g_1+u')(1-e)$$

$$\simeq g_1-u_1'g_1+u'=g_1+u_2'g_2+u_3'g_3 \tag{17}$$

(Remember that primes always mean differentiation with respect to s, not $\tilde{s}$.)

Before we set off to find $\tilde{g}_2$ and $\tilde{g}_3$, we will first go to item 2 listed previously. A generic point A on the track can be specified by the arc length s of the cross section and a location vector ζ in the cross section (Fig. 5). The deformation displacement $u(s,\zeta,t)$ consists of two parts. One is a translation carried along with the center of the cross section—$u(s,t)$; another part is due to the rotation of the cross section. Since the deformations are considered to be small, the small rotations can be treated as vectors. There are two small rotations: one is due to the change of orientation of the centerline tangent g_1, and the other is a rotation around g_1. The new orientation of g_1 is $\tilde{g}_1$; this rotation can be represented by a vector $\alpha=g_1\times\tilde{g}_1$; and the displacement of point A due to this rotation is $\alpha\times\zeta$. Another rotation due to the torsional deformation is $\theta g_1=\Sigma\theta_n\eta_n g_1$; the vector product of this vector and ζ is the corresponding displacement. The whole displacement of point (s,ζ) is

$$u(s,\zeta,t)=u(s,t)+(g_1\times\tilde{g}_1)\times\zeta+\theta g_1\times\zeta \tag{18}$$

Substituting in expression (17) for $\tilde{g}_1$, noting $\zeta=\zeta_2g_2+\zeta_3g_3$, and omitting the second-order terms of η_n, we obtain

$$u(s,\zeta,t)=\sum\psi_n\eta_n+(u_2'g_3-u_3'g_2)\times(\zeta_2g_2+\zeta_3g_3)$$
$$+\sum\Theta_n\eta_n y'\times\zeta=\sum_{n=1}^{NM_k}[\psi_n-(\zeta\cdot\psi_n')y'+\Theta_n y'\times\zeta]\eta_n$$

We now define the modal functions of every point on the track:

$$\Psi_n(s,\zeta)=\psi_n-\zeta\cdot\psi_n'y'+\Theta_n y'\times\zeta, \quad n=1,\ldots,NM_k \tag{19}$$

As ψ_n, Θ_n, and y' are known functions of s, $\Psi_n(s,\zeta)$ can easily be calculated. Since for the track (unlike the muff) s and ζ are independent variables, Ψ_n is not time dependent, we simply have

$$u(s,\zeta,t)=\Sigma\Psi_n\eta_n \tag{20}$$

$$\dot{u}(s,\zeta,t)=\Sigma\Psi_n\dot{\eta}_n \tag{21}$$

$$\ddot{u}(s,\zeta,t)=\Sigma\Psi_n\ddot{\eta}_n \tag{22}$$

These quantities are needed to calculate the contributions of the track to the generalized forces.

Let us return to item 1; it is now easy to find $\tilde{g}_2$ and $\tilde{g}_3$ from Eq. (18). Noting that $\tilde{g}_i-g_i=u(s,g_i,t)-u(s,t)$, $i=2$, 3, we can substitute g_2 or g_3 for ζ in Eq. (18) and use Eq. (17) for $\tilde{g}_1$ to find

$$\tilde{g}_2=g_2+(u_2'g_3-u_3'g_2)\times g_2+\theta g_1\times g_2=g_2-u_2'g_1+\theta g_3 \tag{23}$$

$$\tilde{g}_3=g_3+(u_2'g_3-u_3'g_2)\times g_3+\theta g_1\times g_3=g_3-u_3'g_1-\theta g_2 \tag{24}$$

Together with $\tilde{g}_1$ in Eq. (17), $\tilde{g}_i$ can be written in matrix form

$$\begin{bmatrix}\tilde{g}_1\\ \tilde{g}_2\\ \tilde{g}_3\end{bmatrix}=\begin{bmatrix}1 & u_2' & u_3'\\ -u_2' & 1 & \theta\\ -u_3' & -\theta & 1\end{bmatrix}\begin{bmatrix}g_1\\ g_2\\ g_3\end{bmatrix} \tag{25}$$

or $\{\tilde{g}\}=[B]\{g\}$, where { } represents a column matrix. Matrix $[B]$, as shown in Eq. (25), obviously represents an infinitesimal rotation; the corresponding rotation vector is $\gamma=\alpha+\theta g_1=\theta g_1-u_3'g_2+u_2'g_3$. The deformed base vectors can also be expressed as

$$\tilde{g}_i = g_i + \gamma \times g_i, \qquad i=1,2,3$$

After these preparations, we can now derive the important kinematic quantities—partial (angular) velocities. First, the velocity of h_k in frame $b_{c(k)}$, when it slides along the track with the muff, is

$$V^{h_k} = \overset{\circ}{\tilde{y}} = \tilde{y}'\dot{s} + \frac{\partial \tilde{y}}{\partial t} = (y'+u')\dot{s} + \Sigma \psi_n \dot{\eta}_n \tag{26}$$

Second, to obtain the angular velocity $\tilde{\omega}$ of the frame $\tilde{g}_i$ relative to frame $b_{c(k)}$ when its origin h_k slides along the track with the muff, differentiate Eq. (25) with respect to time in frame $b_{c(k)}$:

$$\{\overset{\circ}{\tilde{g}}\} = [B]\{\overset{\circ}{g}\} + [\dot{B}]\{g\} \tag{27}$$

Note

$$\overset{\circ}{g}_i = \omega_k g_j - \omega_j g_k, \qquad \overset{\circ}{\tilde{g}}_i = \tilde{\omega}_k \tilde{g}_j - \tilde{\omega}_j \tilde{g}_k$$

where $i=1,2,3$ and i,j,k form a positive cycle of 1,2,3.

$$\begin{bmatrix} \overset{\circ}{\tilde{g}}_1 \\ \overset{\circ}{\tilde{g}}_2 \\ \overset{\circ}{\tilde{g}}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_3\tilde{g}_2 - \tilde{\omega}_2\tilde{g}_3 \\ \tilde{\omega}_1\tilde{g}_3 - \tilde{\omega}_3\tilde{g}_1 \\ \tilde{\omega}_2\tilde{g}_1 - \tilde{\omega}_1\tilde{g}_2 \end{bmatrix} = \begin{bmatrix} 1 & u_2' & u_3' \\ -u_2' & 1 & \theta \\ -u_3' & -\theta & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \kappa g_2 \\ \tau g_3 - \kappa g_1 \\ -\tau g_2 \end{bmatrix} \dot{s} + \begin{bmatrix} 0 & \dot{u}_2' & \dot{u}_3' \\ -\dot{u}_2' & 0 & \dot{\theta} \\ -\dot{u}_3' & -\dot{\theta} & 0 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

Noting that

$$\dot{u}_i' = \frac{\mathrm{d}}{\mathrm{d}t}(g_i \cdot u') = \overset{\circ}{g}_i \cdot u' + g_i \cdot \left(u''\dot{s} + \sum \psi'\dot{\eta}\right), \qquad i=2,3$$

we have

$$\dot{u}_2' = (-\kappa u_1' + \tau u_3' + u_2'')\dot{s} + \sum \psi_{n2}'\dot{\eta}_n$$

$$\dot{u}_3' = (-\tau u_2' + u_3'')\dot{s} + \sum \psi_{n3}'\dot{\eta}_n$$

and

$$\dot{\theta} = \sum \Theta_n'\eta_n\dot{s} + \sum \Theta_n\dot{\eta}_n$$

where $\psi_{ni}' = g_i\psi_n'$. Note that

$$\tilde{\omega}_i = -\tilde{g}_j \cdot \overset{\circ}{\tilde{g}}_k = \tilde{g}_k \cdot \overset{\circ}{\tilde{g}}_j \qquad (i,j,k \text{ form a positive cycle})$$

and from Eq. (25), we know that $\tilde{g}_j \cdot g_k$ is the element B_{jk} of matrix $[B]$. After some calculation and omitting the second-order terms of small quantities, and adding a term $(\tan\tilde{\alpha}/R \times \dot{s} = (\tan\alpha/R)\dot{s} = \nu\dot{s}$ to $\tilde{\omega}_1$, we finally obtain the components of angular velocity of frame b_k (embedded in the muff) relative to frame $b_{c(k)}$:

$$\tilde{\omega}_1 = (\tau + \nu + \kappa u_3' + \theta')\dot{s} + \sum \Theta_n\dot{\eta}_n$$

$$\tilde{\omega}_2 = (\kappa\theta - u_3'')\dot{s} - \sum \psi_{n3}'\dot{\eta}_n$$

$$\tilde{\omega}_3 = [\kappa(1-u_1') + u_2'']\dot{s} + \sum \psi_{n2}'\dot{\eta}_n \tag{28}$$

and

$$\tilde{\omega}^{kc(k)} = \tilde{\omega}_1\tilde{g}_1 + \tilde{\omega}_2\tilde{g}_2 + \tilde{\omega}_3\tilde{g}_3$$

Now from Eqs. (28), (26), and (21), one can easily write all the anticipated partial (angular) velocities corresponding to the generalized speeds $\dot{s}$ and $\dot{\eta}$ (see Fig. 2). If we take $\dot{s}$ as the pth generalized speed, then

$$\omega_p^j = \begin{cases} (\tau + \nu + \kappa u_3' + \theta')\tilde{g}_1 + (\kappa\theta - u_3'')\tilde{g}_2 \\ \mp [\kappa(1-u_1') + u_2'']\tilde{g}_3, \text{ for } j \in P(k) \\ 0, \text{ otherwise} \end{cases}$$

$$V_p^{h_j} = \begin{cases} y' + u') + \omega_p^j \times X^{kj}, \text{ for } j \in P(k) \\ 0, \text{ otherwise } V_p^{\eta_j} = 0 \end{cases} \tag{29}$$

If we take $\dot{\eta}_n$ as the pth generalized speed, then

$$\omega_p^j = \begin{cases} \Theta_n\tilde{g}_1 - \psi_{n3}'\tilde{g}_2 + \psi_{n2}'\tilde{g}_3, \text{ for } j \in P(k) \\ 0, \text{ otherwise} \end{cases}$$

$$V_p^{h_j} = \begin{cases} \psi_n + \omega_p^j \times X^{kj}, \text{ for } j \in P(k) \\ 0, \text{ otherwise} \end{cases}$$

$$V_p^{\eta_j} = \begin{cases} \Psi_n(s,\zeta), \text{ for } j=k \\ 0, \text{ otherwise} \end{cases} \tag{30}$$

The above $V_p^{\eta_j}$ are the partial velocities of point (s,ζ) corresponding to $\dot{s}$ and $\dot{\eta}_n$ respectively.

In the case of a straight, unscrewed, flexible track, which may have much practical interest in engineering, α, ν, τ, and κ equal zero, so the forms of Eqs. (28) and (29) will be simpler.

Track on Outboard Body

A typical hinge with a curved flexible track on the outboard body is shown in Fig. 6. If there is another body q outboard of body k, let p_q be the link point leading to it; otherwise, let p_q be the mass center of body k. Since p_q is determined by the structure, it cannot be selected. However, h_k is now a point selected as the origin of the vectors y and $\tilde{y}$; it is also the origin of frame b_k. Since body k is flexible, it only has one point, p_q, that is fixed in frame b_k. Point h_k is generally not fixed in the body. The deformation models ψ_n are measured in frame b_k, and p_q is their fixed point. In this case, the definitions of y and

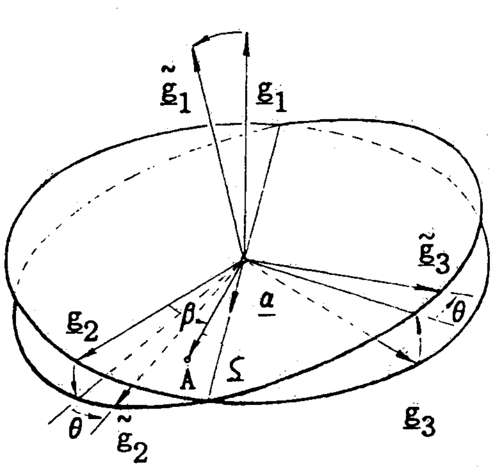

Fig. 5 Differential orientational changes of a cross section.

$\tilde{y}$ are opposite to that shown in Fig. 2. But the relations

$$\tilde{y}(s,t) = y(s) + \Sigma\psi_n(s)\eta_n(t)$$

$$\{\tilde{g}\} = [B]\{g\}$$

are the same.

Kinematic relations have relative means. It is wise not to derive the motion of frame b_k relative to frame b_{pk}, which is fixed on the muff, and first to do the reverse work. The motion of the muff in frame b_k has already been shown in Eqs. (26) and (28). Now, when the frame b_{pk} is supposed to be stationary, we will find the angular velocity of frame b_k, $\tilde{\omega}^*$, and the velocity of h_k in frame b_{pk}. It is obvious that $\tilde{\omega}^*$ is just the negative vector represented by Eq. (28):

$$\tilde{\omega}^* = -(\tilde{\omega}_1 g_1 + \tilde{\omega}_2 \tilde{g}_2 + \tilde{\omega}_3 \tilde{g}_3) \tag{31}$$

Thus all of the partial angular velocities are the same as those in Eqs. (29) and (30), only with opposite sign.

Suppose p_k^* is a point of frame b_k coinciding with p_k at the present time. Obviously,

$$V^{p_k^*} \text{ (relative to frame } b_{pk}) = -V^{p_k} \text{ (relative to frame } b_k)$$

and from Eq. (26): $V^{p_k} = \tilde{y}'\dot{s} + \Sigma\psi\dot{\eta}$. Thus, we have (see Fig. 6)

$$V^{h_j} = V^{p_k^*} + \tilde{\omega}^* \times Z^{kj} = -(\tilde{y}'\dot{s} + \Sigma\psi\dot{\eta}) + \tilde{\omega}^* \times Z^{kj} \tag{32}$$

where Z^{kj} is the vector leading from p_k to h_j, and specifically $Z^{kk} = -\tilde{y}(s,t)$. Thus, the partial velocities of h_j are just the same as those in Eqs. (29) and (30), only with opposite sign and use Z^{kj} instead of X^{kj}.

As for the partial velocities of any point on the track, see Fig. 7. A generic point of the track can be specified by (ξ,ζ), and $z(s,\xi,\zeta,t) = \tilde{y}(\xi,t) + \zeta - \tilde{y}(s,t)$ is the vector from the origin of frame b_{pk} to point (ξ,ζ), or A. Let us take frame b_{pk} as a stationary base and frame b_k as a moving base, and divide the velocity of A into two parts: the velocity of A^*, which is a point of the moving base coinciding with point A at the moment, and the relative velocity of A in the moving base. According to Eqs. (32) and (21),

$$\begin{aligned} V_A &= -(\tilde{y}'\dot{s} + \Sigma\psi\dot{\eta}) + \tilde{\omega}^* \times z + \dot{\hat{u}}(\xi,\zeta) \\ &= -\tilde{y}'\dot{s} + \Sigma[\Psi_n(\xi,\zeta) - \psi_n(s)]\dot{\eta}_n + \tilde{\omega}^* \times z \end{aligned} \tag{33}$$

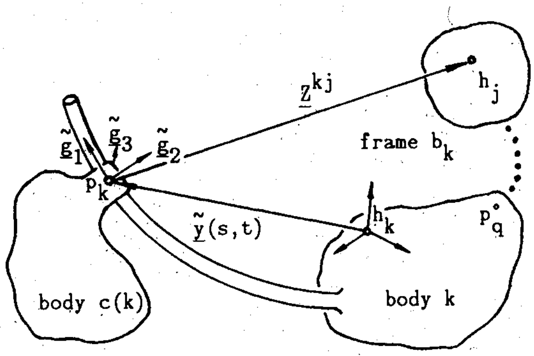

Fig. 6 Curve track on outboard body.

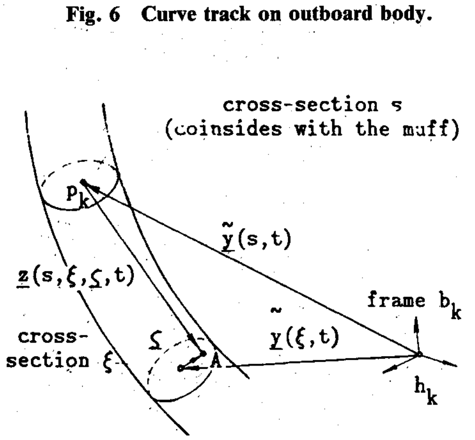

Fig. 7 Geometric relations on outboard track.

From Eqs. (31-33), we can finally write the necessary partial (angular) velocities. If we take $\dot{s}$ as the pth generalized speed, then

$$\omega_p^j = \begin{cases} -\{(\tau + \nu + \kappa u_3' + \theta')\tilde{g}_1 + (\kappa\theta - u_3'')\tilde{g}_2 \\ + [\kappa(1 - u_1') + u_2'']\tilde{g}_3\}, \text{ for } j \in P(k) \\ 0, \text{ otherwise} \end{cases}$$

$$V_p^{hj} = \begin{cases} -(y' + u') + \omega_p^j \times Z^{kj}, \text{ for } j \in P(k) \\ 0, \text{ otherwise} \end{cases}$$

$$V_p^{\eta j} = \begin{cases} = -(y' + u') + \omega_p^j \times z, \text{ for } j = k \\ 0, \text{ otherwise} \end{cases} \tag{34}$$

If we take $\dot{\eta}_n$ as the pth generalized speed, then

$$\omega_p^j = \begin{cases} -\Theta_n \tilde{g}_1 + \psi_{n3}' \tilde{g}_2 - \psi_{n2}' \tilde{g}_3, \text{ for } j \in P(k) \\ = 0, \text{ otherwise} \end{cases}$$

$$V_p^h = \begin{cases} -\psi_n + \omega_p^j \times Z^{kj}, \text{ for } j \in P(k) \\ 0, \text{ otherwise} \end{cases}$$

$$V_p^{\dot{\eta} j} = \begin{cases} \Psi_n(\xi,\zeta) - \psi_n(s) + \omega_p^j \times z, \text{ for } j = k \\ 0 \end{cases} \tag{35}$$

Calculation of the Generalized Inertial Forces f_p^*

This problem will be discussed here because it is needed for the track, but these results can also be used to calculate the contribution to f_p^* of any beamlike structures that can be considered as fulfilling the assumptions adopted here for the track, so that cross sections remain undeformed and perpendicular to the centerline of the beam. Our calculation is based on Eq. (15) in Ref. 1. It is convenient to repeat that equation here:

$$\begin{aligned} -f_p^* = \sum_{k=1}^{NB} \Big\{ & m_k(\ddot{R}^{h_k} + \ddot{l}) \cdot V_p^{h_k} + (\dot{H}^{h_k} + m_k l \times \ddot{R}^{h_k}) \cdot \omega_p^k \\ & + \ddot{R}^{h_k} \cdot \int V_p^{\dot{\eta}k} \mathrm{d}m + \dot{\omega} \cdot \int (r + u) \times V_p^{\dot{\eta}k} \mathrm{d}m \\ & + 2\omega \cdot \int \dot{u} \times V_p^{\dot{\eta}k} \mathrm{d}m + \int \overset{\circ\circ}{u} \cdot V_p^{\dot{\eta}k} \mathrm{d}m - \omega \cdot \underline{D}_p^k \cdot \omega \Big\} \end{aligned} \tag{36}$$

This is the most general formulation and it can treat any deformation models. What follows is an example that illustrates how to apply the general expression to those more restricted models of modal deformation given in Eqs. (12) and (13) for a curved track that is undergoing stretch, bending, and torsion.

In Eq. (36), $\mathrm{d}m$ is the mass of an element that is infinitesimally small in all three dimensions. The selection of integral variables is determined by the shape of the cross section. If it is circular, it will be convenient to express ζ as: $\zeta = \zeta(\cos\beta g_2 + \sin\beta g_3)$, if it is rectangular: $\zeta = \zeta_2 g_2 + \zeta_3 g_3$; we will take the former as an example. Suppose ρ is the mass density

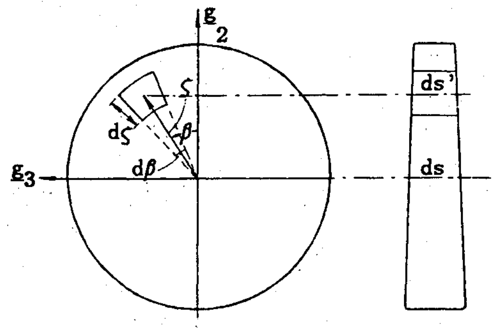

Fig. 8 Differential element on a curved track.

of the track in the undeformed state and Fig. 8 shows an infinitesimal element in an undeformed configuration. Let $a = 1/\kappa$ be the radius of curvature of the undeformed track and ds' be the arc length of our element. From geometric analysis, we have

$$\frac{ds'}{ds} = \frac{a - \boldsymbol{\zeta} \cdot \boldsymbol{g}_2}{a} = 1 - \kappa \boldsymbol{\zeta} \cdot \boldsymbol{g}_2$$

So

$$dm = \rho \zeta d\beta d\zeta ds' = \rho (1 - \kappa \boldsymbol{\zeta} \cdot \boldsymbol{g}_2) \zeta d\beta d\zeta ds \tag{37}$$

It should be pointed out that the differential element changes its volume after deformation, but the mass it contains does not change.

It is not difficult to show that the additional equations [(16-22)] in Ref. 1 will take the following new forms corresponding to our specific models:

$$\dot{\boldsymbol{H}}^{h_k} = \underline{\boldsymbol{I}}^{h_k} \cdot \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \underline{\boldsymbol{I}}^{h_k} \cdot \boldsymbol{\omega} + m_k \sum_{n=1}^{NM} \{\boldsymbol{h}_n \ddot{\eta}_n$$
$$+ [(\underline{\boldsymbol{M}}_n^T + \underline{\boldsymbol{M}}_n) \cdot \dot{\boldsymbol{\omega}}] \eta_n + 2\underline{\boldsymbol{M}}_n \cdot \boldsymbol{\omega} \dot{\eta}_n + \boldsymbol{\omega}$$
$$\times [\underline{\boldsymbol{M}}_n^T + 2\underline{\boldsymbol{M}}_n \underline{\boldsymbol{M}}_n] \cdot \boldsymbol{\omega} \eta_n \}$$
$$+ \int [\boldsymbol{u} \times \ddot{\boldsymbol{u}} + \boldsymbol{u} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{u}) + 2\boldsymbol{u} \times (\boldsymbol{\omega} \times \dot{\boldsymbol{u}})$$
$$+ \boldsymbol{u} \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{u}))] dm$$

where $\underline{\boldsymbol{M}}^T$ is the transposition dyadic of $\underline{\boldsymbol{M}}$, and

$$\boldsymbol{h}_n = \frac{1}{m_k} \int \boldsymbol{r} \times \boldsymbol{\Psi}_n (\boldsymbol{r}) dm$$

$$\underline{\boldsymbol{M}}_n = \frac{1}{m_k} \int [\boldsymbol{\Psi}_n (\boldsymbol{r}) \cdot \boldsymbol{r} \underline{\boldsymbol{U}} - \boldsymbol{\Psi}_n (\boldsymbol{r}) \boldsymbol{r}] dm$$

and

$$\underline{\boldsymbol{D}}_p^k = \int [(\boldsymbol{r} + \boldsymbol{u}) \cdot \boldsymbol{V}_p^{\eta_k} \underline{\boldsymbol{U}} - (\boldsymbol{r} + \boldsymbol{u}) \boldsymbol{V}_p^{\eta_k}] dm$$

and the mass center of the deformed track is located by

$$\boldsymbol{l} = \frac{1}{m_k} \int (\boldsymbol{r} + \boldsymbol{u}) dm$$

where $\boldsymbol{r} = \boldsymbol{y} + \boldsymbol{\zeta}$. The vectors $\boldsymbol{\omega}_p^k$, $\boldsymbol{V}_p^{h_k}$, $\boldsymbol{V}_p^{\eta_k}$, $\boldsymbol{u}$, $\dot{\boldsymbol{u}}$, $\ddot{\boldsymbol{u}}$ and $\boldsymbol{\Psi}_n$ have already been prepared in Eqs. (29) and (30) [or Eqs. (34) and (35)], (20), (21), (22), and (19), respectively, so all the integrals can extend over the whole range of the track.

For instance, let us select a term $\ddot{\boldsymbol{R}}^{h_k} \cdot \int \boldsymbol{V}_p^{\eta_k} dm$ in Eq. (36) and do the integral. Suppose the track is on the inboard body and the pth generalized speed is $\dot{\eta}_n$. From Eqs. (37), (30), and (19) we have

$$\int \boldsymbol{\Psi}_n (s, \boldsymbol{\zeta}) dm = \iiint (\boldsymbol{\psi}_n - \boldsymbol{\zeta} \cdot \boldsymbol{\psi}_n' \boldsymbol{y}'$$
$$+ \Theta_n \boldsymbol{y}' \times \boldsymbol{\zeta}) \rho (1 - \kappa \boldsymbol{\zeta} \cdot \boldsymbol{g}_2) \zeta d\beta d\zeta ds$$

Assuming that ρ depends only on s, dividing the variables and integrating separately, we have

$$\int \boldsymbol{\Psi}_n (s, \boldsymbol{\zeta}) dm = \pi R^2 \int_0^L \rho \boldsymbol{\psi}_n ds + \frac{\pi}{4} R^4 \int_0^L \rho \kappa \psi_{n2}' \boldsymbol{g}_1 ds$$
$$- \frac{\pi}{4} R^4 \int_0^L \rho \kappa \Theta_n \boldsymbol{g}_3 ds$$

where L is the total arc length of the track and R is the radius of the cross section. Because ρ, κ, $\boldsymbol{\psi}_n$, $\boldsymbol{g}_i$ and Θ_n are all known functions of s, we can integrate them by numerical methods in advance of any simulations over time. All of the other integrals in this section can be integrated in a similar manner.

In case the track is slender enough that the inertia moment of the cross section can be omitted, then the mass of the track can be concentrated on the centerline; then we need not do such complicated integrals, and can just take $dm = \pi R^2 \rho ds$, and let $\boldsymbol{\zeta} = 0$ to obtain

$$\int \boldsymbol{V}_p^{\eta_k} dm = \int \boldsymbol{\psi}_n (s) dm = \pi R^2 \int_0^L \rho \boldsymbol{\psi}_n ds$$

Calculation of the Generalized Active Forces f_p

Our calculation is based on Eq. (14) of Ref. 1:

$$f_p = \sum_{k=1}^{NB} \left(\boldsymbol{M}^{h_k} \cdot \boldsymbol{\omega}_p^k + \boldsymbol{F}^k \cdot \boldsymbol{V}_p^{h_k} + \int \boldsymbol{V}_p^{\eta_k} \cdot d\boldsymbol{f} \right) \tag{38}$$

where $\boldsymbol{M}^{h_k}$ is the moment on B_k with respect to the hinge point h_k of the working forces, and $\boldsymbol{F}^k$ is the force on B_k obtained by summing the working forces. For an inboard flexible track $\boldsymbol{\omega}_p^k$, $\boldsymbol{V}_p^{h_k}$, and $\boldsymbol{V}_p^{\eta_k}$, have already been shown in Eqs. (29) and (30). Suppose the track is made of elastic material without structural damping. Using elastic potential energy U, the last term in Eq. (38) can be expressed as

$$\int \boldsymbol{V}_p^{\eta_k} \cdot d\boldsymbol{f} = \int \rho \boldsymbol{V}_p^{\eta_k} \cdot \boldsymbol{P} dv - \frac{\partial U}{\partial \eta_n} \tag{39}$$

$\boldsymbol{P}$ is the volume force (such as gravity) and dv the volume of an element.

According to the restrictive assumptions applied to the track, every cross section is a rigid plane perpendicular to the tangent of the centerline. This is a very strong assumption, implying that Poisson's ratio μ must equal zero and $\lambda = 2\mu G/(1-2\mu) = 0$, $G = E/[2(1+\mu)] = E/2$; and there are no linear strains in the ζ, β directions, and there are also no shear strains in the (ζ, β) and (s, ζ) planes. The only linear strain is $\epsilon_1 (s, \zeta, t)$

$$\epsilon_1 = \frac{d\bar{s}'}{ds'} - 1$$

By the same means of deriving Eq. (16) (using ds' instead of ds), we have the linearized ϵ_1 at point (s, ζ) on the track:

$$\epsilon_1 = \boldsymbol{g}_1 \cdot \boldsymbol{u}' \frac{ds}{ds'} = \boldsymbol{g}_1 \cdot \boldsymbol{u}' / (1 - \kappa \boldsymbol{\zeta} \cdot \boldsymbol{g}_2)$$

Using Eqs. (20) and (19), we obtain

$$\epsilon_1 = \Sigma(g_1 \cdot \psi_n' - \zeta \cdot \psi_n'' + \Theta_n \kappa\zeta \sin\beta)\eta_n / (1 - \kappa\zeta\cos\beta) \tag{40}$$

The only shear strain is

$$\gamma_{13} = \zeta\frac{d\theta}{ds'} = \zeta\frac{d\theta}{ds}\frac{ds}{ds'} = \zeta\Sigma\Theta_i'\eta_i/(1-\kappa\zeta\cos\beta) \tag{41}$$

The potential energy of the track is

$$\begin{aligned} 2U &= \int(\sigma_1\epsilon_1 + \tau_{13}\gamma_{13})dv = \int(E\epsilon_1^2 + G\gamma_{13}^2)dv \\ &= \int\int\int E(\epsilon_1^2 + \tfrac{1}{2}\gamma_{13}^2)(1-\kappa\zeta\cos\beta)\zeta d\beta d\zeta ds \\ &= \int\int\int E\Big\{\{[\Sigma(\psi_{i1}' - \zeta\cdot\psi_i'' + \Theta_i\kappa\zeta\sin\beta)\eta_i]^2 \\ &\quad + \tfrac{1}{2}(\zeta\Sigma\Theta_i'\eta_i)^2\}/(1-\kappa\zeta\cos\beta)\Big\}d\beta d\zeta ds \end{aligned} \tag{42}$$

When we take $\dot{\eta}_n$ as the pth generalized speed, then

$$\begin{aligned} -\frac{\partial U}{\partial \eta_n} &= -\sum_{i=1}^{NM_k}\eta_i\int\int\int E\{[(\psi_{i1}' - \zeta\cdot\psi_i'' \\ &\quad + \Theta_i\kappa\zeta\sin\beta)(\psi_{n1}' - \zeta\cdot\psi_n'' \\ &\quad + \Theta_n\kappa\zeta\sin\beta) + \tfrac{1}{2}\zeta^2\Theta_i'\Theta_n']/(1-\kappa\zeta\cos\beta)\}\zeta d\beta d\zeta ds \end{aligned} \tag{43}$$

Since the factor $(1-\kappa\zeta\cos\beta)$ is in the denominator, it is impossible to separate the variables. But, as assumed previously, the radius R of the cross section is small as compared with the radius ($a = 1/\kappa$) of curvature of the track centerline, $\kappa\zeta \ll 1$, and

$$1/(1-\kappa\zeta\cos\beta) \simeq 1 + \kappa\zeta\cos\beta$$

Then the variables in Eq. (43) can easily be divided and integrated separately (assuming that E depends only on s), as in the previous section.

The generalized active forces then follow from the combination of Eqs. (43) and (39) into Eq. (38).

Conclusion

With the specific expressions derived herein for the generalized forces and with the detailed derivations of the coefficients of generalized speeds ("partial velocities") that form the nucleus of this paper, progress with the problem of describing the dynamics of systems of interconnected flexible bodies has been advanced to include relative interbody translation on curved tracks, whether rigid or flexible.

For these results to have practical utility they must be incorporated into existing formulations of system equations. Both the general equations and the computer codes are in the public domain, so this extension of simulation capability is accessible to all.

References

[1]Singh, R., VanderVoort, R., and Likins, P., "Dynamics of Flexible Bodies in Tree Topology—A Computer Oriented Approach," *Journal of Guidance, Control and Dynamics,* Vol. 8, No. 5, Sept.-Oct. 1985, pp. 584–590.

[2]Bodley, C., Devers, A., Park, A., and Frisch, H., "A Digital Computer Program for Dynamic Interaction and Simulation of Controls and Structures (DISCOS)," NASA TP 1219, May 1978.

[3]Huston, R.L., "Multi-Body Dynamics Including the Effects of Flexibility and Compliance," *Computers and Structures,* Vol. 14, No. 5–6, 1981, pp. 443–451.

[4]Singh, R. and Likins, P., "Manipulator Interactive Design with Interconnected Flexible Elements," *Proceedings of the 1983 American Control Conference,* San Francisco, CA, June 1983, pp. 505–512.

[5]Singh, R. and Likins, P., "Singular Value Decomposition for Constrained Dynamical Systems," *Journal of Applied Mechanics,* Vol. 52, No. 4, 1985, pp. 943–948.

[6]VanderVoort, R. and Singh, R., "A Pointing Control System Simulation and Generic Model Generator for Multi-Body Flexible Structures," *Proceedings of the 9th DARPA Strategic and Space Symposium,* Monterey, CA, Oct. 1983.

[7]Kane, T.R., Likins, P., and Levinson, D., "Complex Spacecraft," *Spacecraft Dynamics,* 1st ed., McGraw-Hill, New York, 1983, Chap. 4, pp. 247–269.